

Backdoors for SAT

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EMCL / TUD

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- 1 Introduction
- 2 Backdoors
- 3 Parameterized Complexity
- 4 Experimental Results

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- Introduced by Williams et al. ([8]) to try to explain the good performances of modern SAT solvers.
 - A backdoor is a set of variables. Once we assign a value to the backdoor variables, the problem becomes tractable (in P)
 - Their claim: modern SAT solver can find backdoors easily.
- But solvers are NOT designed for this!
- ! Why don't we try to pro-actively find these backdoors?
 - Finding backdoors is an NP-Hard problem!

Theoretical work:

- Defining several types of backdoors,
- Complexity of finding them (especially parameterized complexity)

Empirical work:

- Showing that backdoor sets are “small”, for different types of backdoors
 - Results are mostly based on local search algorithms! (Incompleteness)
- Little information on *runtime* required to find them
- One work ([5]) shows that using Horn-backdoors improves SAT solving speed. However, no information on how long it takes to find them!

Questions:

- How can we find backdoors *efficiently*?
- Can we “predict” backdoors by using additional domain knowledge?

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A few auxiliary definitions:

Class \mathcal{C} : a set of formulas sharing some property.

Horn class: A CNF formula F is in Horn iff each of its clauses has *at most* one positive literal.

Subsolver C : A polynomial-time algorithm that can solve all formulas for a given class \mathcal{C}

There are solvers for different classes: Horn, 2SAT, RHorn, UP+PL etc.

Deletion Backdoors (1/2)

$F - B$: Replacing in each clause of F the occurrences of x and $\neg x$ with \perp for each variable $x \in B$ (with $B \subseteq \text{var}(F)$) and simplifies the clause. Basically, remove the occurrences (positive and negative) of the variables in B from F

Definition: Deletion C -backdoor ([4])

A non-empty subset B of the variables of the formula F ($B \subseteq \text{var}(F)$) is a *deletion backdoor* w.r.t. a class C for F iff $F - B \in C$.

Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee z \vee w) \wedge (y \vee w) \quad B = \{z, y\}$$
$$F - B \equiv (x \vee \perp \vee \perp) \wedge (\neg x \vee \perp \vee w) \wedge (y \vee w)$$

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Deletion Backdoors (2/2)

From a deletion C -backdoor we can generate $2^{|B|}$ formulas that can be solved in polynomial time!

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- For an NP-Hard problem we have an exponential worst-case runtime in the size of the problem: e.g. SAT $O(2^n)$
- We can do better by defining a *parameter* k on which to confine the exponential explosion: e.g. p-SAT $O(2^k * n^c)$, with k number of variables and some constant c
- Problems for which we can identify such parameters are called *Fixed-parameter tractable* (FPT)
- Note that SAT parameterized by the size of a backdoor is FPT ($O(2^{|B|} * n^c)$)

Vertex cover (1/2)

k-Vertex cover is one of the best known FPT problems.

Definition: Vertex Cover

Given a graph $G = (V, E)$, we call $C = \{v_1, \dots, v_n\} \subseteq V$ a *vertex cover* of G iff for all $e \in E$ there exists a $v_i \in C$ s.t. $v_i \in e$.

In other words, we have a vertex $v_i \in C$ as *representative* of each edge in E . We call $|C|$ the size of the vertex cover.

Definition: k-Vertex Cover

Instance: A graph $G = (V, E)$

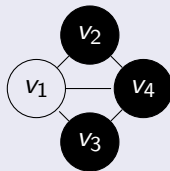
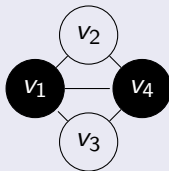
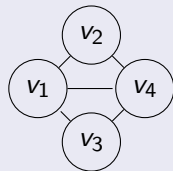
Parameter: Upper-bound on the size of the cover k

Question: Does G have a vertex cover of size $\leq k$?

Vertex Cover (2/2)

Example

Consider $G = (V, E)$:



Then $C_1 = \{v_1, v_4\}$ and $C_2 = \{v_2, v_3, v_4\}$ are vertex cover for G but only C_1 is a solution for the k -Vertex cover instance $(G, 2)$.

Reduction to Vertex Cover (1/2)

Samer and Szeider ([7]) propose a reduction from deletion Horn-backdoor detection to Vertex Cover.

Definition

G_F is the graph composed by the variables of the CNF formula F in which two variables v, u are adjacent iff v and u appear positively in a clause from F .

Lemma

A set $B \subseteq \text{var}(F)$ is a deletion Horn-backdoor for F iff B is a vertex cover of G_F

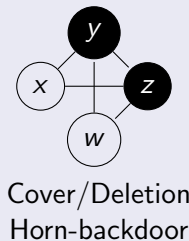
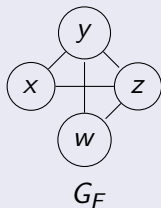
This relation extends also to Minimum Vertex Cover, in which we are interested in the vertex cover/backdoor of minimal size (smallest backdoor).

Reduction to Vertex Cover (2/2)

Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee z \vee w) \wedge (y \vee w)$$

$$F - C \equiv x \wedge (\neg x \vee w) \wedge w$$



We can use existing results from Vertex Cover (including FPT results) to solve deletion Horn-backdoor detection!

- Study deletion Horn-backdoors in SAT instances
- Build a dataset *efficiently*, e.g. FPT
 - No available implementation of parameterized Vertex Cover
- Implement algorithms that are a good trade-off between performance and implementation complexity
- Test whether we can use local search to efficiently find smallest deletion Horn-backdoors
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- FPT algorithms: Kernelization and Bounded search
- Local search algorithm: COVER ([6])
- Reduction to SAT

Kernelization

Given a parameterized problem (x, k) , a kernelization is a polynomial time preprocessing technique that either:

- returns a new equisatisfiable instance (x', k') (with $|x'| \leq g(k)$ and $k' \leq k$),
- rejects the instance as unsatisfiable

The high degree kernelization is simple but powerful:

HdK A vertex of degree $> k$ must be in any cover of size $\leq k$.

By applying some other pre-processing to our graph, we can apply the following result:

Property ([1])

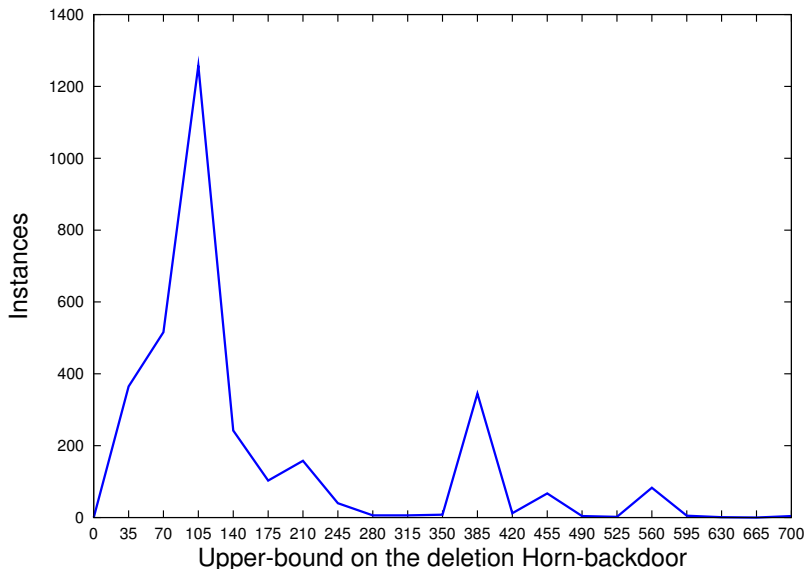
If G is a graph with a vertex cover of size k and there is no vertex of G with degree $> k$ or degree < 3 , then $|V| \leq \frac{k^2}{3} + k$.

- We implement a simple bounded search by Hüffner ([3]),
- The *trivial* solution for vertex cover has complexity $O(2^k)$
- The *best* $O(1.2738^k)$
- This one $O(1.47^k)$

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- 1 Benchmark with 3239 instances from various sources;
- 2 Generate the associated vertex cover instances;
- 3 Run a modified version of COVER [6] to obtain an upper-bound on the size of the smallest deletion Horn-backdoor;
- 4 For instances with small backdoors ($k \leq 150$) verify the minimality of the backdoor;
- 5 For instances with bigger backdoors confirm that the lower-bound is bigger than 150;
- 6 Considering only the instances for which we have the exact value of the smallest backdoor, compute the quality of the solution provided by a fast version of COVER.

Results (1/3)



2418 (74%) instances have upper-bound on the size of the deletion Horn-backdoor up to 150:

- 2357 (97.5%) verified by the FPT algorithms,
- 8 more with CryptoMinisat,
- 53 remain unverified

Runtime¹

- Time-out: 90 minutes
- Average: 25 second; 87% in less than 5 seconds, 93% in under a minute

The generated vertex cover instances were in most of the cases easy, but a few were really hard (2%).

¹Timings based on Intel Centrino 1.7Ghz, 1GB RAM

Kernelization is important:

- 42% of all instances were solved by kernelization (51.5% in the group with $k \leq 150$);
- Otherwise, parameter reduction of 17.8% (avg)
- Parameter has exponential influence on runtime
- Extreme case from $k = 109$ to $k' = 6$

219 instances might have a solution of size ≤ 150 , but remain unverified.

COVER Results

COVER builds a greedy solution by picking the k vertices with highest degree. Iteratively, it tries to find a smaller solution by exchanging “randomly” vertices that are in the cover with other vertices.

Full computation:

- 10^5 attempts/iterations, try to improve by 10%
- Runtime: 30-90 minutes
- Solution quality: Always finds the optimum

Fast computation:

- 10^4 attempts/iterations, exit as soon as no solution is found for a given k'
- Runtime: 115 ms (avg), 97 ms (avg) for $k \leq 150$
- Solution quality: 98% of the times optimal

Average error among all the instances is 0.11%.

COVER is a good method to compute smallest deletion Horn-backdoors.

Correlations

We are interested in the relation between smallest deletion Horn-backdoor (sdH-bd) size and other properties of the instances:

- Flat** Colouring on flat graphs; sdH-bd size is exactly two times the number of vertices in the graph.
- Random** Uniform random formulas (uf/uuf): correlation between number of variables and sdH-bd size is 0.99
- CarConf** Verify some consistency properties of requested car configuration. Correlation between number of variables and upper-bound of the sdH-bd size for the *same* configuration is high:

Base configuration	Correlation (r)	# Instances
C168	0.99	58
C170	0.99	6
C202	0.83	23
C208	0.99	16
C210	0.89	32
C220	0.95	348
C638	0.73	84

We studied the relation between deletion Horn-backdoor detection and vertex cover.

- Kernelization is fundamental! Even with a simple kernelization, we solved many instances without search.
- COVER is a good way of computing quickly (and with an excellent quality) smallest deletion Horn-backdoors
- Domain information of the instances can be related to its backdoors (or at least to the size of the smallest deletion Horn-backdoor)

SAT:

- Implement COVER/deletion Horn-backdoor detection in a solver and allow branching only on backdoors variables;
- Use backdoors to explore different solver architectures, and not only DPLL;
- Influence of preprocessing on different classes of backdoors;
- Build predictive models that can find backdoors!

Parameterized complexity:

- Implement, compare and release more algorithms: Chen's vertex cover, or Abu-Khzam parallel vertex cover;
- Redo the work presented here for another class that is also FPT: e.g. deletion RHorn or deletion Cluster.

Questions?

Datasets, tools and slides are available online: <http://marco.gario.org/work/>



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- F : a propositional formula in CNF
- $var(F)$: the set of variables occurring in F .
- l, \bar{l} : a positive variable or its negation
- J : (partial) interpretation. Partial mapping from $var(F)$ to the boolean values $\{\top, \perp\}$.
- Class \mathcal{C} : a set of formulas sharing some property.
- Horn class: F is in Horn iff each of its clauses has *at most* one positive literal.

Example

- $F = a \wedge \neg b \wedge (\neg d \vee c) \wedge (\neg c \vee d)$
- $\text{var}(F) = \{a, b, c, d\}$
- $J = \{\bar{b}, c\}$
- $F|_J \equiv a \wedge \cancel{\neg b} \wedge (\cancel{\neg d} \vee \cancel{c}) \wedge (\cancel{\neg c} \vee d) \equiv a \wedge d$
- For $V = \{c\}$,
 $F - V \equiv a \wedge \neg b \wedge (\neg d \vee \cancel{c}) \wedge (\cancel{\neg c} \vee d) \equiv a \wedge \neg b \wedge \neg d \wedge d$
- $F \in \text{Horn}$

Definition: Subsolver [8]

We call an algorithm C a subsolver if, given an input formula F :

Tricotomy: C either rejects the input F , or “determines” F correctly (as unsatisfiable or satisfiable, returning a solution if satisfiable),

Efficiency: C runs in polynomial time,

Trivial solvability: C can determine if F is trivially true (has no constraints) or trivially false (has contradictory constraint),

Self-reducibility: if C determines F , then for any assignment J of the variable x C determines $F|_J$

There exists a subsolver for Horn! And also for other classes: e.g. RHorn, 2SAT, UP+PL.

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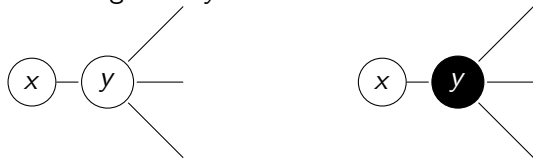
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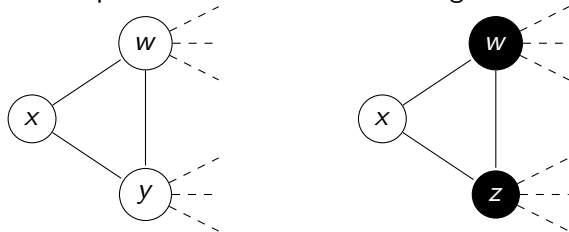
For preprocessing we use the following four rules:

- P1** A vertex of degree 0 cannot be part of any cover, therefore we obtain G' by removing it from G .
- P2** If there is a vertex x of degree 1, then there is an optimal vertex cover in which its neighbour y is in the cover.

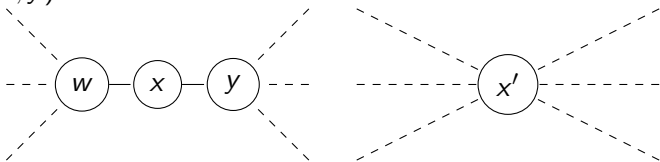


Simple Preprocessing II

P3 If there is a vertex of degree 2 with two adjacent neighbours, then there is an optimal vertex cover containing both these neighbours.

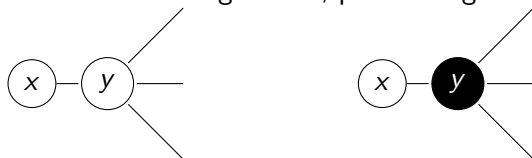


P4 If there is a vertex x of degree 2 with two non-adjacent neighbours y and w , then x can be removed by contracting the edges (w, x) and (x, y) .



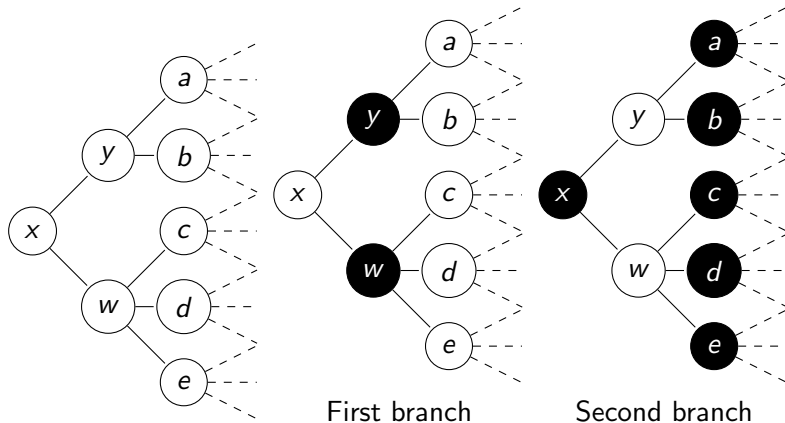
We use this simple bounded search by Hüffner ([3])

S1 If there is a vertex of degree one, put its neighbour into the cover.



S2 If there is a vertex x of degree two, then either i) both neighbours of x are in an optimal vertex cover, or ii) x is in an optimal cover together with all neighbours of its neighbours.

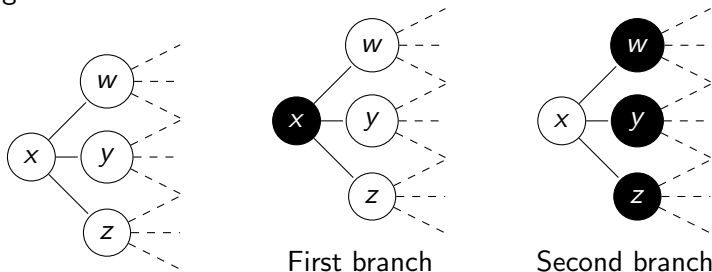
Bounded search II



First branch

Second branch

- S3 If there is a vertex x of degree at least three, then either x or all its neighbours are in the cover.



Strong/Deletion Backdoors (1/2)

$F|_J$: reduct of F w.r.t. the (partial) interpretation J ; it is obtained by replacing each variable v in F with $J(v)$ and simplifying.

Definition: Strong C -Backdoor ([8])

A non-empty subset B of the variables of the formula F ($B \subseteq \text{var}(F)$) is a *strong backdoor* w.r.t. the subsolver C for F iff **for all** interpretations $J : B \rightarrow \{\top, \perp\}$, C returns a satisfying assignment or concludes unsatisfiability of $F|_J$.

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Strong/Deletion Backdoors (1/2)

For some classes, like Horn, deletion backdoor and strong backdoor are equivalent. ([2])

Example

This is good! To verify whether B is a deletion \mathcal{C} -backdoor we just need to check whether $F - B$ is in \mathcal{C} and not whether all the $2^{|B|}$ reducts of F are.

Local Search: COVER

